# Limits and Derivatives

## Introduction

- Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change.

## Limits

- In general as x → a, f(x) → l, then l is called limit of the function f(x)
- Symbolically written as $\lim_{x \to a} f(x) = l$.
- For all the limits, function should assume at a given point x = a
- The two ways x could approach a number an either from left or from right, i.e., all the values of x near a could be less than a or could be greater than a.
- The two types of limits
  - Right hand limit
    - Value of f(x) which is dictated by values of f(x) when x tends to from the right.
  - Left hand limit.
    - Value of f(x) which is dictated by values of f(x) when x tends to from the left.
- In this case the right and left hand limits are different, and hence we say that the limit of f(x) as x tends to zero does not exist (even though the function is defined at 0).

## Algebra of limits

**Theorem 1**

Let f and g be two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

- Limit of sum of two functions is sum of the limits of the function s,i.e
  $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$
- Limit of difference of two functions is difference of the limits of the functions, i.e.
  $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
- Limit of product of two functions is product of the limits of the functions, i.e.,
  $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x).$
- Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e.,
  $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
- In particular as a special case of (iii), when g is the constant function such that g(x) = λ, for some real number λ, we have
  $\lim_{x \to a} [(\lambda . f)(x)] = \lambda . \lim_{x \to a} f(x)$.

## Limits of polynomials and rational functions

- A function f is said to be a polynomial function if f(x) is zero function or if f(x) = $a_0 + a_1 x + a_2 x^2 + . . . + a_n x^n$, where $a_i$S is are real numbers such that an ≠ 0 for some natural number n.

- We know that $\lim_{x \to a} x = a.$

  $$\lim_{x \to a} x^2 = \lim_{x \to a}(x.x) = \lim_{x \to a} x . \lim_{x \to a} x = a.\, a = a^2$$

  Hence,

  $$\lim_{x \to a} x^n = a^n$$

- Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a polynomial function

  $$\lim_{x \to a} f(x) = \lim_{x \to a}\left[a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n\right]$$

  $$= \lim_{x \to a} a_0 + \lim_{x \to a} a_1 x + \lim_{x \to a} a_2 x^2 + \ldots + \lim_{x \to a} a_n x^n$$

  $$= a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \ldots + a_n \lim_{x \to a} x^n$$

  $$= a_0 + a_1 a + a_2 a^2 + \ldots + a_n a^n$$

  $$= f(a)$$

- A function f is said to be a rational function, if f(x) = $\frac{g(x)}{h(x)}$ where g(x) and h(x) are polynomials such that h(x) ≠ 0.
  Then

  $$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}$$

- However, if h(a) = 0, there are two scenarios –
  - when g(a) ≠ 0
    - limit does not exist
  - When g (a) = 0.
    - g(x) = $(x - a)^k g_1(x)$, where k is the maximum of powers of (x – a) in g(x)
    - Similarly, h(x) = $(x - a)^l h_1(x)$ as h (a) = 0. Now, if k ≥ l, we have

$$\lim_{x \to a} f(x) = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{\lim_{x \to a} (x-a)^k g_1(x)}{\lim_{x \to a} (x-a)^l h_1(x)}$$

$$= \frac{\lim_{x \to a} (x-a)^{(k-l)} g_1(x)}{\lim_{x \to a} h_1(x)} = \frac{0.g_1(a)}{h_1(a)} = 0$$

If k < l, the limit is not defined.

**Theorem 2**

For any positive integer n

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

**Proof**

Dividing $(x^n - a^n)$ by $(x - a)$, we see that

$$x^n - a^n = (x-a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x\,a^{n-2} + a^{n-1})$$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x\,a^{n-2} + a^{n-1})$$

$$= a^{n-1} + a\,a^{n-2} + \ldots + a^{n-2}(a) + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1} \ (n \text{ terms})$$

$$= na^{n-1}$$

**Note:**

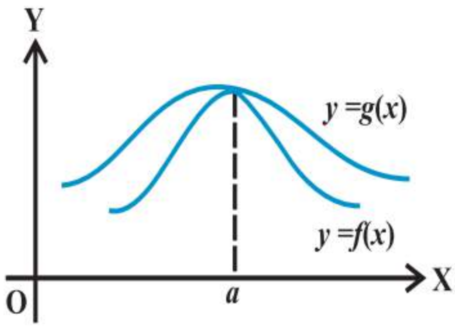
The expression in the above theorem for the limit is true even if n is any rational number and a is positive.

## Limits of Trigonometric Functions

**Theorem 3**

Let f and g be two real valued functions with the same domain such that f(x) ≤ g(x) for all x in the domain of definition,

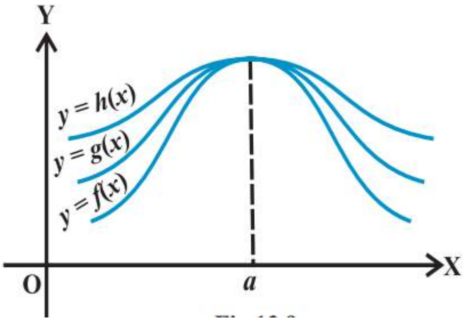
For some a, if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$




**Theorem 4 (Sandwich Theorem)**

Let f, g and h be real functions such that f(x) ≤ g(x) ≤ h(x) for all x in the common domain of definition.

For some real number $a$, if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l.$

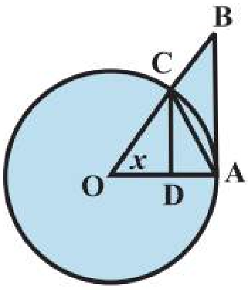



**To Prove:**

$$\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < |x| < \frac{\pi}{2}$$

**Proof:**

We know that sin (– x) = – sin x and cos (– x) = cos x. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$




- is the centre of the unit circle such that the angle AOC is x radians and $0 < x < \frac{\pi}{2}$
- Line segments B A and CD are perpendiculars to OA. Further, join AC. Then
- Area of ΔOAC < Area of sector OAC < Area of Δ OAB

i.e., $\frac{1}{2}\text{OA.CD} < \frac{x}{2\pi}.\pi.(\text{OA})^2 < \frac{1}{2}\text{OA.AB}$.

i.e., CD < $x$ . OA < AB.

From $\Delta$ OCD,

$\sin x = \frac{\text{CD}}{\text{OA}}$ (since OC = OA) and hence CD = OA sin $x$. Also $\tan x = \frac{\text{AB}}{\text{OA}}$ and

hence AB = OA. tan $x$. Thus

OA sin $x$ < OA. $x$ < OA. tan $x$.

Since length OA is positive, we have

$\sin x < x < \tan x$.

Since $0 < x < \frac{\pi}{2}$, sin$x$ is positive and thus by dividing throughout by sin $x$, we have

$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$. Taking reciprocals throughout, we have

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$$\cos x < \frac{\sin x}{x} < 1$$

Hence Proved

The following are two important limits

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$. (ii) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

**Proof:**

The function $\frac{\sin x}{x}$ is sandwiched between the function cos x and the constant function which takes value 1.

Since $\lim_{x \to 0} \cos x = 1$, also we know that $1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\sin\left(\frac{x}{2}\right)$$

$$= \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\lim_{x \to 0} \sin\left(\frac{x}{2}\right) = 1.0 = 0$$

Using the fact that x → 0 is equivalent to $\frac{x}{2} \to 0$. This may be justified by putting y= $\frac{x}{2}$

## Derivatives

- **Some Real time Applications**
  - People maintaining a reservoir need to know when will a reservoir overflow knowing the depth of the water at several instances of time
  - Rocket Scientists need to compute the precise velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times.
  - Financial institutions need to predict the changes in the value of a particular stock knowing its present value.
  - Helpful to know how a particular parameter is changing with respect to some other parameter.
- Derivative of a function at a given point in its domain of definition.
- **Definition 1**
  - Suppose f is a real valued function and a is a point in its domain of definition.
  - The derivative of f at a is defined by

    $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ Provided this limit exists.
  - Derivative of f(x) at a is denoted by f'(a)
- **Definition 2**
  - Suppose f is a real valued function, the function defined by

    $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ Wherever limit exists is defined to be derivative of f at x
    - Denoted by f'(x).
    - This definition of derivative is also called the **first principle of derivative.**

    Thus $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
    - f'(x) is denoted by $\frac{d}{dx}(f(x))$ or if y = f(x), it is denoted by dy/dx.
    - This is referred to as derivative of f(x) or y with respect to x.
    - It is also denoted by D (f(x)).
    - Further, derivative of f at x = a

    is also denoted by $\left.\frac{d}{dx} f(x)\right|_a$ or $\left.\frac{df}{dx}\right|_a$ or even $\left(\frac{df}{dx}\right)_{x=a}$.

**Theorem 5**

- Let f and g be two functions such that their derivatives are defined in a common domain. Then
  - Derivative of sum of two functions is sum of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$

  - Derivative of difference of two functions is difference of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x).$$

- Derivative of product of two functions is given by following product rule.

$$\frac{d}{dx}\left[f(x)\ .\ g(x)\right]=\frac{d}{dx}f(x).g(x)+f(x).\frac{d}{dx}g(x)$$

- Derivative of quotient of two functions is given by the following quotient rule (whenever the denominator is non–zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}f(x).g(x)-f(x)\frac{d}{dx}g(x)}{(g(x))^2}$$

- Let u =f (x) = and v = g (x).
  - **Product Rule:**
    - (uv) ′ = u' v+ uv′.
    - Also referred as Leibnitz rule for differentiating product of functions
  - **Quotient rule**
    - $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$

- Derivative of the function f(x) = x is the constant

**Theorem 6**

- Derivative of f(x) = $x^n$ is $nx^{n-1}$ for any positive integer n.
- **Proof**
  - By definition of the derivative function, we have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}.$$

Binomial theorem tells that $(x+h)^n=\left({}^nC_0\right)x^n+\left({}^nC_1\right)x^{n-1}h+...+\left({}^nC_n\right)h^n$ and hence $(x+h)^n-x^n=h(nx^{n-1}+...+h^{n-1})$. Thus

$$\frac{df(x)}{dx}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{h\left(nx^{n-1}+....+h^{n-1}\right)}{h}$$

$$=\lim_{h\to 0}\left(nx^{n-1}+...+h^{n-1}\right),=nx^{n-1}.$$

  - This can be proved as below alternatively

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(x.x^{n-1})$$

$$= \frac{d}{dx}(x).(x^{n-1}) + x.\frac{d}{dx}(x^{n-1}) \text{ (by product rule)}$$

$$= 1.x^{n-1} + x.((n-1)x^{n-2}) \text{ (by induction hypothesis)}$$

$$= x^{n-1} + (n-1)x^{n-1} = nx^{n-1}.$$

**Theorem 7**

- Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + .... + a_1 x + a_0$ be a polynomial function, where $a_i$s are all real numbers and $a_n \neq 0$. Then, the derivative function is given by

$$\frac{df(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1}x^{x-2} + ... + 2a_2 x + a_1.$$

**Quick Reference:**

- **For functions f and g the following holds:**

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to\infty} g(x)}$$

- **Following are some of the standard limits**

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \lim_{x\to a}\frac{\sin(x-a)}{x-a} = 1$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0}\frac{\tan x}{x} = 1, \lim_{x\to a}\frac{\tan(x-a)}{x-a} = 1$$

$$\lim_{x\to 0}\frac{\sin^{-1}x}{x} = 1, \lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$$

$$\lim_{x\to 0}\frac{a^x - 1}{x} = \log_e a, a > 0, a \neq 1$$

- **Derivatives**
  - The derivative of a function f at a is defined by

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

- Derivative of a function f at any point x is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

- **For functions u and v the following holds:**

$$(u \pm v)' = u' \pm v'$$

$$(uv)' = u'v + uv' \qquad \Rightarrow \qquad \frac{d}{dx}(uv) = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad \Rightarrow \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

- **Following are some of the standard derivatives**

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\cos ec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x.\tan x$$

$$\frac{d}{dx}(\cos ecx) = -\cos ecx.\cot x$$